An Overview To The Excluded Grid Theorem

Seminar for Algorithmic Graph Theory

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This paper summarizes the proceeding of the so called Excluded Grid Theorem of Robertson and Seymour. It states the existence of a bound of treewidth for a graph to not contain a grid of specified size as a minor. While recent research in this topic improved far forward we focus on a recent paper from Julia Chuzhoy[4] showing a new improved and simplified proof, work out key concepts and techniques used by her and other referenced publications.

1 Introduction

Global assumptions about graphs is a wide field of research in graph theory. One way to express those are graph properties. Using these it is possible to classify groups of graphs and therefore adapt to those customizing algorithms or argue about possible relations.

Definition 1 (Graph Property)**.** *If $\mathcal{H}$ is any set or class of graphs, then the class $Forb_{\preceq}(\mathcal{H}) := \{G|G \not\succeq H : H \in \mathcal{H}\}$ of all graphs without a minor in $\mathcal{H}$ is a graph property, here expressed with forbidden minors.[5, pp. 263]*

In this paper we focus on a classification of linkedness within a graph specified by a grid.

Theorem 1 (Excluded Grid Theorem)**.** *There is some function $f : Z \mapsto Z^+$, such that for any integer $g \geq 1$, any graph of treewidth at least $f(g)$ contains the $(g \times g)$-grid as a minor.[7]*

1986 Robertson & Seymour published an article including the Excluded Grid Theorem[7], shown in theorem 1. Hence it is possible to express a class of graphs with bounded tree-width of $f(g)$ using a graph property by definition 1 and choosing $\mathcal{H} = \{(g \times g)\}$ respectively. This is in turn used to optimize algorithms like it is done in research for the disjoint paths problem from Neil Robertson and Paul D Seymour[9].

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While research in tree-width continues, finding a closer bound for $f(g)$ gets more important. 2014 Chekuri and Chuzhoy[2] were first providing a polynomial upper bound for f with $f(g) = O\left(g^{98}\text{poly}\log g\right)$. Further improvements were published by Julia Chuzhoy[4] to $f(g) = O\left(g^{36}\text{poly}\log g\right)$ in year 2015. In addition she simplified the proof introducing a new iterative algorithm to obtain a so called path-of-sets system, which was already used by Chekuri and Chuzhoy[2] before. This system again is used to gain a minor grid as shown by Chandra Chekuri and Julia Chuzhoy in [1] and is closely related to a structure used by Alexander Leaf and Paul Seymour in [6] called grill.

This paper will mostly refer to Julia Chuzhoy's[4] improved and simplified version of proofing the Excluded Grid Theorem. In order to refer to previous work skipping explanations the following sections will give a more complete overview about techniques and shared notation used within the proof as a whole taken from multiple publications. Thus explanations will be more informal and present the key aspects in a more informal and intuitive way at the cost of depth and integrity. As a result this paper is a short introduction and overview to the Excluded Grid Theorem and deeper understanding can be obtained from the referenced publications.

2 Preliminaries

2.1 Basic Notation

Path

A path is a sequence of edges $P_k = (\{v_1, w_1\}, \ldots, \{v_k, w_k\})$ with $w_i = v_{i+1}$ for all $1 \leq i < k$, $\{v_1, w_1\}, \{v_k, w_k\}$ are called endpoints and all others inner vertex of P_k. For convenience a path is written as a set of edges and their ordering is assumed to be well defined. $V(P) := \bigcup_{e \in P} e$ is denoted as the vertices of P and $E(P) := \bigcup_{e \in P} \{e\}$ as the edges of P, accordingly $E(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} E(P)$ and $V(\mathcal{P}) = \bigcup_{p \in \mathcal{P}} V(P)$ for a set of paths $\mathcal{P}$. For every path P, P is called internally disjoint from a set of vertices $V_0 \subseteq V$ if $V(P) \cap V_0 = \emptyset$ and internally disjoint from any other path P' if every $v_0 \in V(P) \cap V(P')$ is an endpoint of both paths or furthermore node disjoint if $V(P) \cap V(P') = \emptyset$. Let $\mathcal{P}$ be any set of paths in G, $\zeta : \mathfrak{P} \mapsto \mathbb{N}, \zeta(\mathcal{P}) := \max_{P_0 \in \mathcal{P}} \max_{e_0 \in P_0} \{\sum_{P \in \mathcal{P}} |e_0 \cap P|\}$ is the maximum number of edges shared by all paths and $\eta_{\mathcal{P}} = \zeta(\mathcal{P})$ is named the caused edge-congestion of $\mathcal{P}$.

Flow

For any two subsets $S, T \subseteq V$ of vertices a set of paths $F : S \rightsquigarrow T$(short F) is called a flow from source S to tear T if for every vertex $s_o \in S$ exists a unique path $P_o \in F$ containing s_0 as endpoint in S and any endpoint $t_0 \in T$. If all endpoints of all paths in F are distinct F is written $F : S \overset{k:k}{\rightsquigarrow} T$ with $k = |F|$, for $k = |S| = |T|$, F can be written as $F : S \overset{1:1}{\rightsquigarrow} T$. Additionally F can be written as $F : S \rightsquigarrow_\eta T$ respectively $F : S \overset{1:1}{\rightsquigarrow}_\eta T$ with η is the maximum edge-congestion F can cause. A flow F is node disjoint if all paths within F are node disjoint. Notice that for every node disjoint flow F the caused edge-congestion is $\eta_F = 1$.

2.2 Minor

Let G_1, G_2 be undirected graphs. G_1 is a minor of G_2, written $G_1 \preceq G_2$, if G_1 can be obtained by a sequence of predefined operations, listed below, applied to G_2.

$$\begin{aligned}
\alpha : \mathcal{G} \times E \mapsto \mathcal{G}, \alpha((V,E),\{v,w\}) \quad &:= (V \setminus \{w\}, E \setminus \{\{u,w\}|u \in V\} \cup \{\{u,v\}|\{u,w\} \in E \wedge u \neq v\}) \\
\beta : \mathcal{G} \times E \mapsto \mathcal{G}, \beta((V,E),e) \quad &:= (V, E \setminus e) \\
\gamma : \mathcal{G} \times V \mapsto \mathcal{G}, \gamma((V,E),v) \quad &:= (V \setminus \{v\}, E)
\end{aligned}$$

α is called edge contraction, β edge deletion and γ node deletion. Notice the order of performed operation does not effect the resulting graph. Figure 1 shows an example of a minor. Additionally we define

$$\begin{aligned}
&\alpha : \mathcal{G} \times \mathcal{E} \mapsto \mathcal{G}, \quad \alpha(G,E) := \alpha(\alpha(G,e_0), E \setminus \{e_0\}), \text{ for any } e \in E \text{ if } E \neq \emptyset \text{ else } \alpha(G,E) := G \\
&\alpha : \mathcal{G} \times \mathcal{P} \mapsto \mathcal{G}, \quad \alpha(G,P) := \alpha(G,E(P))
\end{aligned}$$

the contraction of sets of edges and the contraction of paths.

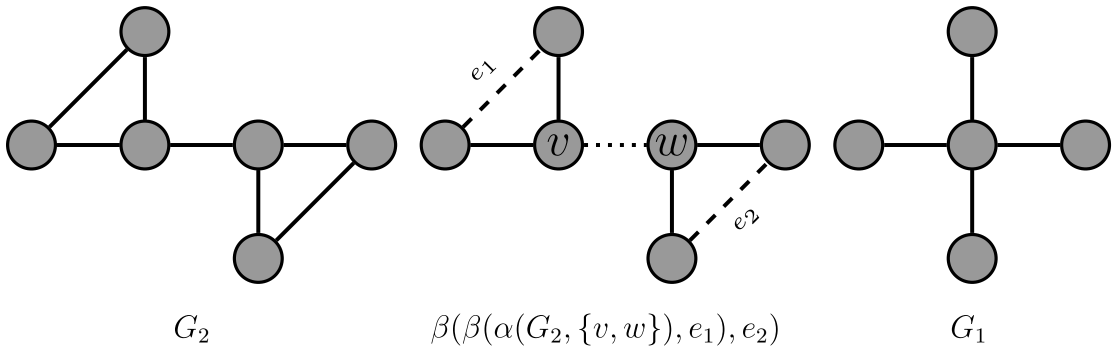

Figure 1: An example of a minor $G_1 \preceq G_2$. The sequence of operations (α, β, β) to obtain G_1 from G_2 is illustrated in the middle. The dashed lines show an edge deletion and the dotted line an edge contraction.

2.3 Linkedness

Linkedness measures connectivity within a graph in context of edges or nodes. While node connectivity is strongly related to treewidth it is often more practical to work with edge connectivity. We will start explaining the linkedness of edges and later derive node-linkedness from edge-linkedness under certain circumstances.

Definition 2 (α-well-linkedness)**.** *Let $G = (V,E)$ be any undirected graph and $V_0 \subseteq V$ a subset of vertices. We call V_0 α-well-linked if $|E(A,B)| \geq \alpha * \min\{|A \cap V_0|, |B \cap V_0|\}$ holds for any partition (A,B) of G and $0 < \alpha \leq 1$.[1, Definition 2.1]*

Definition 2 uses edge-cuts and hence defines edge-well-linkedness, using flows 3 expresses an alternative definition.

Definition 3. *Let $G = (V, E)$ be any undirected graph and $V_0 \subseteq V$ a subset of vertices. We call V_0 α-well-linked if for all subsets $V_1, V_2 \subseteq V_0$ with $|V_1| = |V_2|$ and $V_1 \cap V_2 = \emptyset$ exists a Flow $F : V_1 \overset{1:1}{\rightsquigarrow} V_2$ in G.[4, Definition 2.1]*

In addition a set of vertices $V_0 \in V$ is (k', α)-well-linked if definition 3 holds for all subsets with cardinality at most k' respectively $|E(A, B)| \geq \alpha * \min\{|A \cap V_0|, |B \cap V_0|, k\}$ for definition 2.

Definition 4 (node-well-linkedness)**.** *For any undirected graph $G = (V, E)$, let $V_0 \subseteq V$ a subset of vertices. We call V_0 node-well-linked in G if for any two subsets $V_1, V_2 \subseteq V_0$ with $V_0 = V_1$ and $V_0 \cap V_1 = \emptyset$ exists a node disjoint flow $F : V_1 \overset{1:1}{\rightsquigarrow}_1 V_2$ in G.*

To derive node-well-linkedness from edge-well-linkedness we assume that for a given undirected graph $G = (V, E)$ the maximum degree of all vertices is bounded by $\max_{v \in V}\{\delta(v)\} \leq 3$. This infers for any unequal two paths $P_1, P_2 \in F : S \overset{1:1}{\rightsquigarrow}_1 T$ in G every node $v \in V(P_1) \cap V(P_2)$ is either endpoint in P_1 or P_2. To exclude this case we further assume for every vertex $v \in S \cup T$ the degree is bounded by $\delta(v) \leq 1$. The strong relation between node-well-linkedness and treewidth illustrates the following lemma.

Lemma 1. *Let k be the size of the largest node-well-linked vertex set in G. Then $k \leq tw(G) \leq 4k$.[8]*

2.4 Degree Reduction

To derive node-linkedness from edge-linkedness we fixed some assumptions. These hold using a sub-graph G' that satisfy the requirements. The existence of such sub-graph is given by the following theorem.

Theorem 2. *Let G be any graph of treewidth k. Then there is a sub-graph G' of G, whose maximum vertex degree is 3, and $tw(G) = \Omega(k/\text{poly}\log k)$. Moreover, there is a set $T \subseteq V(G')$ of $\Omega(k/\text{poly}\log k)$ vertices, such that T is 1-well-linked in G, and each vertex of T has degree 1 in G.[3]*

3 Path-of-Sets System

The path-of-sets system is the main combinatorial object in both [1] and [4]. It is used as intermediate step to obtain a grid minor from a given graph.

Definition 5 (path-of-sets system)**.** *Let $G = (V, E)$ be any undirected graph. We call $\mathbb{P}_{r,h} = (\mathcal{S}, \mathcal{F})$ a path-of-sets system with width r and height h of G under following conditions.*

- *The sequence $\mathcal{S} = \{S_1, \ldots, S_r\} \subseteq \mathfrak{P}(V)$ is disjoint and for every $1 \leq i \leq r$, $G[S_i]$ is connected.*
- *For the set of flows $\mathcal{F} = \left\{F_1 : S_1 \overset{h:h}{\rightsquigarrow}_1 S_2, \ldots, F_{r-1} : S_{r-1} \overset{h:h}{\rightsquigarrow}_1 S_r\right\}$, all paths in $\mathcal{P} = \bigcup_{F \in \mathcal{F}}$ are node disjoint and there is no $F_i \in \mathcal{P}$ containing a vertex $v_0 \in \bigcup_{S_i \in \mathcal{S}} S_i$ as inner vertex.*
- *For any $1 < i < r$ the sets of vertices $A_i = S_i \cap V(F_{i-1})$ and $B_i = S_i \cap V(F_i)$ are 1-linked in $G[D_i]$.*

[1, p. 9]

Julia Chuzhoy uses a more strict definition of 3 in [4, Definition 2.7]. Unfortunately it is not needed and therefore left out. In addition she distinguish between different path-of-sets systems based on the linkedness of A_i and B_i. In this context definition 3 is categorized as strong path-of-sets system. Other categories are weaker definitions and a strong path-of-sets can be obtained from them to the cost of height.

Theorem 3. *Every graph G containing a path-of-sets system $\mathbb{P}_{h,h}$ minor contains the $(\sqrt{h} \times \sqrt{h})$-grid as a minor as well.*

Let $\mathbb{P}_{h,h} = (\mathcal{S}, \mathcal{F})$ be a path-of-sets system in a graph $G = (V, E)$ according to its definition. For every $1 < i < h$ let $Q_i = F : A_i \overset{h:h}{\rightsquigarrow} B_i$, that exists because (A_i, B_i) is 1-linked by definition. Furthermore $V(Q_i) \subseteq S_i$ and S_i is connected. Hence for every path $P_1 \in Q_i$ exists at least on path $P_2 \in Q_i$ connected to P_1 threw another path $\beta_{P_1P_2} \subseteq S_i$ with no inner vertex in $V(Q_i)$.

The proof of theorem 3 in [1] uses this structure to build a new set of paths $\mathcal{H}$ by concatenating $F_1Q_2F_2, \ldots, Q_{r-1}F_{r-1}$. Informal, the paths in $\mathcal{H}$ can be considered as the horizontal lines of the grid. The vertical lines are obtained by rerouting the horizontal paths using the specified interconnections $\beta_{P_1P_2}$ within the sets from $\mathcal{S}$. Notice because $\beta_{P_1P_1} \subseteq S_i$ is internally disjoint from all other paths in Q_i contraction of $\beta_{P_1P_2}$ will not affect the way P_1 and P_2 traverse $\mathcal{H}$. Building a vertical path of length $\sqrt{h}$ can cause the contraction of $\sqrt{h}$ sets. As a result $\mathbb{P}$ contains a $(\sqrt{h} \times \sqrt{h})$-grid as a minor. This is far away from a proof and in fact more complicated but shows the general idea. For the complete more in depth proof see [1, p. 50].

With regard to theorem 3 it is sufficient to proof every undirected graph G with $\text{tw}(G) \geq f(g)$ contains a $\mathbb{P}_{g^2,g^2}$ path-of-sets system as a minor to proof the excluded grid theorem 1.

4 Splitting Clusters

This section is the main contribution of [4]. It shows an iterative approach to split a path-of-sets system $\mathbb{P}^i_{r_i,h_i}$ to double its width while shrinking the height $h_{i+1} = h/2^{17(i+1)}$[4, Theorem 3.1]. To create a minor $(g \times g)$-grid width and height of g^2 are acquired, according to theorem 3. The iterative process start with an initial path-of-sets system of width 1. Hence $2\log_2 g$ steps are performed to get sufficient width and therefore initial height of $h_0 = 2g^{36}$ is required. Combined with the degree reduction with a cost of $\text{poly}\log g$ in treewidth we get the result $f(g) = O(g^{36}\text{poly}\log g)$ from Julia Chuzhoy's paper [4].

Let $G = (V, E)$ be the graph from a performed degree reduction, described earlier, with terminals $T \subset V$. T is by definition 1-linked therefore $\mathbb{P}^1_{1,2g^{36}} = (\{T\}, \{F : T_1 \rightsquigarrow_1 T_2\})$ is a valid initial path-of-sets system, with $T_1, T_2 \subset T$ are 2 disjoint and equal sized subsets of terminals. Notice G is connected and so are all paths $P, P' \in \mathbb{P}^1$.

To split a path-of-sets system every cluster $S \in \mathcal{S}$ will be split separately. Therefore S splits into two disjoint clusters $C_1, C_2 \subseteq S \setminus (T_1 \cup T_2)$, well-linked in $G[C_1]$ and $G[C_2]$ respectively. Additionally both clusters are required to be connected to T_1 by a flow $F : C_1 \cup C_2 \overset{k:k}{\rightsquigarrow}_1 T_1$. This is then called a weak 2-cluster chain, a more formal definition is found in [1, Definition 4.1]. More important is the existence of a 2-cluster chain.

Definition 6 (2-cluster chain)**.** *Let G be a graph, T_1, T_2 two disjoint sets of vertices, with $|T_1| = k$ and $|T_2| = k' = k/64$, where $k \geq 12000$ is a power of 2. A 2-cluster chain $(X, Y, \tilde{T_1}, \tilde{T_2}, E')$ consists of:*

- *two disjoint clusters $X, Y \subseteq V(G)$*
- *a subset $\tilde{T_1} \subseteq T_1 \cap X$, with $|\tilde{T_1}| = k'$, and a subset $\tilde{T_2} \subseteq T_2 \cap Y$, with $|\tilde{T_2}| = k/512$*
- *a set $E' \subseteq E(X, Y)$ of $k/512$ edges, whose endpoints are all distinct*

Let $\Upsilon_X \subseteq X$ be the subset of vertices of X incident on the edges of E', and let $\Upsilon_Y \subseteq Y$ be the subset of vertices of Y incident on the edges of E'. Then:

- *$\tilde{T_1} \cup \Upsilon_X$ is $(k/512, a^*)$-well-linked ind $G[X]$ and $\tilde{T_2} \cup \Upsilon_Y$ is $(k/512, a^*)$-well-linked in $G[Y]$, for $a^* = 1/64$.*

[4, Definition 3.1]

Considering the sequence $(Y_1, X_1, \ldots, Y_r, X_r)$ the 2-cluster chain is strongly related to the path-of-sets system. Therefore every cluster, created by a split operation, needs to contain a 2-cluster chain. These can be obtained from a weak 2-cluster chain, hence it is sufficient to show their existence within a cluster created by a split operation.

To proof the existence of a weak 2-cluster chain, the clusters are handled differently based on their connectivity to their surrounding environment. Therefore Julia Chuzhoy uses the bandwidth property and balanced cuts(see [4, p. 5]) showing that linkedness of the terminals can be used to argue about the connectivity of clusters. Informally said, nodes of a cluster connected to the terminals with disjoint paths can be connected together using their internally linkage. These must exist as G is connected and has an appropriate treewidth. With a good partition of the terminals T the structure repeats itself again with every iterative step and choosing an appropriate height of terminals for $\mathbb{P}^0$ delivers a sufficient path-of-sets system $P^{2 \log g}_{g^2, g^2}$ containing a $(g \times g)$-grid as a minor.

5 Conclusion

The proof of the excluded grid theorem has several approaches. All of these share a common projection of linkedness and its relation to treewidth. Though all share similar techniques and combinatorial objects, Julia Chuzhoy delivers a more lightweight and simplified proof while improving the bound. It was very interesting to see how treewidth, node-linkedness and edge-linkedness can be used in such a closely related and exchangeable manner as a structural tool for minors. Her recursive solution shows promising results and we can expect more publications, based on her research, in the near future.

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